Congruences for real algebraic curves on an ellipsoid

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Abstract

The problem of arrangement of a real algebraic curve on a real algebraic surface is related to the 16th Hilbert problem. We prove in this paper new restrictions on arrangement of nonsingular real algebraic curves on an ellipsoid. These restrictions are analogues of Gudkov-Rokhlin, Gudkov-Krakhnov-Kharlamov, Kharlamov-Marin congruences for plane curves (see e.g. [1] or [2]). To prove our results we follow Marin approach [3] that is a study of the quotient space of a surface under the complex conjugation. Note that the Rokhlin approach [4] that is a study of the 2-sheeted covering of the surface branched along the curve can not be directly applied for a proof of Theorem 1 since the homology class of a curve of Theorem 1 can not be divided by 2 hence such a covering space does not exist.

1 Formulations of results

It is well-known that a complex quadric is isomorphic to $\mathbf{C}P^1 \times \mathbf{C}P^1$ and an algebraic curve on a quadric is defined by a bihomogeneous polynomial of bidegree (d, r). If the curve is real and the quadric is an ellipsoid then $d = r$ and the curve can be represented as the intersection of an ellipsoid and a surface of degree d in $\mathbf{R}P^3$ (this is because a curve of bidegree $(d, r), d \neq r$ can not be invariant under the involution of the complex conjugation of ellipsoid).

Let Q be an ellipsoid, $\mathbf{R}Q$ and $\mathbf{C}Q$ be the spaces of its real and complex points; A be a nonsingular real algebraic curve of bidegree (d, d) on Q; $\mathbf{R}A$ and $\mathbf{C}A$ be the spaces of real and complex points of A. Components of $\mathbf{R}A$ are called ovals and the number of ovals of $\mathbf{R}A$ is denoted by l. $\mathbf{R}A$ divides $\mathbf{R}Q$ into two parts with a common boundary. Let B_0 and B_1 denote these parts in such a way that in the case when l is even congruence $\chi(B_0) \equiv 0 \pmod 4$ is correct (as it is usual, χ denotes Euler characteristic).

According to F.Klein, A is a curve of type I (II) if $\mathbf{C}A \setminus \mathbf{R}A$ is not connected (otherwise). It is well-known (see [5]) that if A is a curve of type I then $\mathbf{R}A$ has two natural opposite orientations.

We need the following definitions to formulate Theorem 2. Let us choose one of two complex orientation of $\mathbf{R}A$ and some orientation of B_0. Oval C of $\mathbf{R}A$ is called disorienting if these orientations induce opposite orientations on it. C divides $\mathbf{R}Q$ into two disks D and D'. Let $x(D)$ be equal to $\chi(B_1 \cap D) \bmod 2$. It is clear that if $l \equiv 0 \pmod 2$ then $x(D) = x(D')$; in this case we set $x(C)$ to be equal to $x(D)$.

The main results of this paper are the following.

Theorem 1 *Let d be an odd number.*

a) *If A is an M-curve (i.e. $l = (d-1)^2 + 1$) then*

$$\chi(B_0) \equiv \chi(B_1) \equiv \frac{d^2+1}{2} \pmod 8$$

b) *If A is an (M-1)-curve (i.e. $l = (d-1)^2$) then*

$$\chi(B_0) \equiv \frac{d^2-1}{2} \pmod 8$$

$$\chi(B_1) \equiv \frac{d^2+3}{2} \pmod 8$$

c) *If A is an (M-2)-curve (i.e. $l = (d-1)^2 - 1$) and*

$$\chi(B_0) \equiv \frac{d^2-7}{2} \pmod 8$$

then A is of type I.

d) *If A is of type I then*

$$\chi(B_0) \equiv \chi(B_1) \equiv 1 \pmod 4$$

Theorem 2 *Let d be an even number.*

a) *If A is an M-curve and if all components of B_1 have even Euler characteristics then*

$$\chi(B_0) \equiv d^2 \pmod{16}$$

$$\chi(B_1) \equiv 2 - d^2 \pmod{16}$$

b) *Let A be a curve of type I with chosen complex orientation. If there exist an orientation of B_0 such that $x(C) = 0$ for every disorienting oval C then*

$$\chi(B_0) \equiv d^2 \pmod 8$$

$$\chi(B_1) \equiv 2 - d^2 \pmod 8$$

Remark: Propositions d) of Theorem 1 and b) of Theorem 2 can be deduced from the formula of complex orientations of Zvonilov [6].

2 Proof of Theorem 1

Let W denote $B_0 \cup \mathbf{C}A/conj$. As it was shown in [7] a manifold $\mathbf{C}Q/conj$ where $conj$ is the involution of complex conjugation is diffeomorphic to $\overline{\mathbf{C}P^2}$ It is easy to check that

$$W \circ W = d^2 - 2\chi(B_0) \equiv 1 \pmod 2 \tag{1}$$

Hence, W is a characteristic surface in $\mathbf{C}Q/conj$. Let us apply Guillou-Marin congruence [8]

$$\sigma(\mathbf{C}Q/conj) \equiv W \circ W + 2\beta(q) \pmod{16}$$

where σ is signature, $\beta(q)$ is Brown invariant of quadratic Guillou-Marin form $q : H_1(W; \mathbf{Z}_2) \to \mathbf{Z}_4$ of surface W in $\mathbf{C}Q/conj$. This congruence, equality 1 and equality $\sigma(\overline{\mathbf{C}P^2}) = -1$ follow that

$$\chi(B_0) \equiv \frac{d^2+1}{2} + \beta(q) \pmod 8.$$

The calculations of $\beta(q)$ below are similar to the calculations in [3], [9]. Let L denote the subspace of $H_1(W; \mathbf{Z}_2)$ generated by classes realized by ovals of $\mathbf{R}A$. Let $U = L^\perp$ denote the orthogonal complement of L with respect to the intersection form of W. It is clear that $U^\perp = L \subset U$ and $q|_L = 0$. Therefore, U is an informative subspace and $\beta(q) = \beta(q')$, where q' is the form on $U/U^\perp$ induced by q (see [9]). According to [9] one can get the following.

- **a)** $dim(U/U^\perp) = 0$, hence $\beta(q') \equiv 0 \pmod 8$
- **b)** $dim(U/U^\perp) = 1$, hence $\beta(q') \equiv \pm 1 \pmod 8$
- **c)** $dim(U/U^\perp) = 2$, $\beta(q') \equiv 4 \pmod 8$, hence q' is even, therefore A is of type I
- **d)** q' is even, hence $\beta(q') \equiv 0 \pmod 4$.

3 Lemma for Theorem 2

Let A_+ be the closure of one of two components of $\mathbf{C}A \setminus \mathbf{R}A$. Let W_0 denote $A_+ \cup B_0$ and let W_1 denote $A_+ \cup B_1$.

Lemma 1 *Surface W_1 is not homologous to zero modulo 2 in $\mathbf{C}Q$*

Proof If W_1 is homologous to zero then W_1 is a characteristic surface in $\mathbf{C}Q$ and we can apply Guillou-Marin congruence

$$0 \equiv W_1 \circ W_1 + 2\beta(q_1) \pmod{16}$$

where q_1 is the Guillou-Marin form of surface W_1 in $\mathbf{C}Q$. But $W_1 \circ W_1 = d^2 - \chi(B_1)$ and $\beta(q_1) \equiv 0 \pmod 2$ since $dimH_1(W_1; \mathbf{Z}_2) \equiv \chi(W_1) \equiv 0 \pmod 2$, note that $\chi(W_1)$ is even, because W_1 is the result of gluing of two orientable surfaces along whole boundary. Congruences above and $d^2 \equiv 0 \pmod 4$ imply that $\chi(B_1) \equiv 0 \pmod 4$ follow that $\chi(B_1) \equiv 0 \pmod 4$. This is a contradiction to the choice of B_1.

4 Proof of Theorem 2

Let e_1 and e_2 denote the elements of $H_2(\mathbf{C}Q)$ realized by generating lines of quadric $\mathbf{C}Q$. It is clear that e_1 and e_2 form a basis of $H_2(\mathbf{C}Q)$, $conj_* e_1 = -e_2$, $conj_* e_2 = -e_1$. Let (α, β) denote $\alpha e_1 + \beta e_2, \alpha, \beta \in \mathbf{Z}$. Each of generating lines transversally intersects $\mathbf{R}Q$ at one point, hence $[\mathbf{R}Q] \equiv (1,1) \pmod 2$. From relations $[W_j] - conj_*[W_j] \equiv [\mathbf{C}A] \equiv (d,d) \pmod 2), j = 0,1,$ $[W_0] + [W_1] \equiv [\mathbf{R}Q] \equiv (1,1) \pmod 2$ and Lemma 1 one can deduce that $[W_0] \equiv 0 \pmod 2$. Therefore W_0 is a characteristic surface in $\mathbf{C}Q$ and there is Guillou-Marin form q_0 on $H_1(W_0; \mathbf{Z}_2)$. Value of q_0 on the element of $H_1(W_0; \mathbf{Z})$ realized by oval C is equal to $x(C)$ (for the calculations see [10]). Let L_0 denote the subspace of $H_1(W_0; \mathbf{Z}_2)$ generated by such ovals C that $x(C) = 0$. In the case 2 we have $L_0 = L_0^{\perp}$ therefore $\beta(q_0) \equiv 0 \pmod 8$. In the case 2 we have that form $q_0' : L_0^{\perp} \to \mathbf{Z}_4$ induced by q_0 is even therefore $\beta(q_0) \equiv 0 \pmod 4$. Now theorem 2 follows from Guillou-Marin congruence [8]

$$0 \equiv d^2 - \chi(B_0) + 2\beta(q_0) \pmod{16}$$

5 Applications for the curves of bidegrees (3,3) and (5,5)

Theorem 1 and Harnack inequality give a complete system of restrictions for real schemes of flexible curves of bidegree (3,3) (a definition of flexible curves on an ellipsoid is similar to the definition given in [1] for plane curves). All real schemes realizable by flexible curves of bidegree (3,3) can be realized by algebraic curves of bidegree (3,3) (the classification of real schemes of algebraic curves of bidegree (3,3) is $<\alpha>$, $\alpha \leq 5$ and $1 \sqcup 1<1>$ – see e.g. [6]).

Real schemes of M-curves of bidegree (5,5) allowed by Theorem 1 are $\alpha \sqcup 1<\beta>, \alpha+\beta = 16, \beta \equiv 2 \pmod 4$ and $\alpha \sqcup 1<\beta> \sqcup 1<\gamma>, \alpha+\beta+\gamma = 15, \alpha \equiv 1 \pmod 4$. These schemes except might be $1 \sqcup 1<6> \sqcup 1<8>$ and $1 \sqcup 1<5> \sqcup 1<9>$ can be constructed by smoothing singularities of images under birational transformations of appropriate plane M-curves of degree 5 intersecting a real line at 5 different real points and M-curves of degree 6 transversally intersecting a real line at 4 different real points.

6 An absence of congruences similar to Theorem 1 for curves of even bidegrees

There is no congruence modulo 4 for each halves B_1, B_2 of a complement of M-curve of even bidegree (because in this case $l \equiv 0 \pmod 2$), hence $\chi(B_0) \equiv 0 \pmod 2$ and $\chi(B_0) \equiv \chi(B_1) + 2 \pmod 4$). A classification of the real schemes of curves of bidegree (4,4) shows an absence of nontrivial congruences similar to Theorem 1 : all schemes allowed by Harnack inequality and Bezout theorem are realizable by algebraic curves - $<\alpha>$,$\alpha \leq 10$ and $\alpha \sqcup 1<\beta>$, $\alpha + \beta \leq 9$ (see e.g.[6]).

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A Congruences for curves with imaginary singularities

This appendix can be regarded as the addendum to the paper of V.M.Kharlamov and O.Ya.Viro [9]. From theorems of paper [9] under appropriate birational transformations one can deduce the congruences of Matsuoka [11] for curves on a hyperboloid. But the congruences of these paper can not be deduced in this way, because images of nonsingular curves on an ellipsoid under birational isomorphism have imaginary singular points. There are no consideration of curves with imaginary singular points in [9] , but the approach of [9] can be applied to these curves either. Let us consider this application.

Link L in S^3 is said to be even if the linking number of each component of L and the set of all other component is even. The Arf-invariant of even link L is equal by the definition to the Arf-invariant of Seifert form of some Seifert surface of L.

Let A be the plane real curve of degree $m = 2k$ and let $\mathbf{R}P^2_+$ be one of two parts of $\mathbf{R}P^2$ bounded by $\mathbf{R}A$. We shall say that a singular point of $\mathbf{C}A$ is even if the link of this singular point is even. We shall set the Arf-invariant of singular point to be equal to the Arf-invariant of the link of this singular point.

A.1 *Suppose that all singular points of A are imaginary and even. Let Ar be the sum of Arf-invariants of singular points taken per one from each pair of complex-conjugated singular points. Let $\mathbf{R}P^2_+$ be orientable.*

- **a)** *If A is an M-curve then* $\chi(\mathbf{R}P^2_+) \equiv k^2 + 4Ar \pmod 8$
- **b)** *If A is an (M-1)-curve then* $\chi(\mathbf{R}P^2_+) \equiv k^2 \pm 1 + 4Ar \pmod 8$
- **c)** *If A is an (M-2)-curve of type II then* $\chi(\mathbf{R}P^2_+) \equiv k^2 + d + 4Ar \pmod 8$*, where* $d \in \{0, 2, -2\}$
- **d)** *If A is of type I then* $\chi(\mathbf{R}P^2_+) \equiv k^2 \pmod 4$

The proof of Theorem A.1 is similar to the proofs of theorems in [9].

Theorem 1 of present paper can be deduced from the application of A.1 to plane curves of degree $2d$ with two imaginary complex-conjugated nondegenerate singular points of multiplicity d with the help of A.4 (it is easy to see that a nondegenerate singular point of multiplicity d is even iff d is odd).

Let us extend theorem 3.A of [9] to the case of singular curves with imaginary singular points (for the notations in A.2 see [9]).

A.2 *Suppose that $\mathbf{Z}_4$-quadratic form $\tilde{Q_\Delta}$ is informative and suppose that all imaginary singular points of A are even.*

- **a)** *If A is an M-curve then* $\chi(\mathbf{R}P^2_+) \equiv k^2 + \beta(\tilde{q_\Delta}) + \tilde{b} + 4Ar \pmod 8$
- **b)** *If A is an (M-1)-curve then* $\chi(\mathbf{R}P^2_+) \equiv k^2 \pm 1 + \beta(\tilde{q_\Delta}) + \tilde{b} + 4Ar \pmod 8$
- **c)** *If A is an (M-2)-curve of type II then* $\chi(\mathbf{R}P^2_+) \equiv k^2 + d + \beta(\tilde{q_\Delta}) + \tilde{b} + 4Ar \pmod 8$*, where* $d \in \{0, 2, -2\}$
- **d)** *If A is of type I then* $\chi(\mathbf{R}P^2_+) \equiv k^2 + \beta(\tilde{q_\Delta}) + \tilde{b} \pmod 4$

Example 1 *Let us consider the application of A.2 to the problem of apparent contour of real cubic surface. Let* $\mathbf{R}B$ *be a surface of degree 3 in* $\mathbf{R}P^3$*. Point* $a \in \mathbf{R}P^3$ *determine projection* $pr : \mathbf{R}B \to \mathbf{R}P^2$*. The image of singular point of this projection is called the apparent contour of cubic. A generic apparent contour of cubic is a real plane singular curve with 6 singular points – cusp points (this can be deduced from the fact that an apparent contour is determined by the discriminant of cubic polynomial).*

A.3 *An apparent contour of cubic surface can not have real schemes represented on figures 1 and 2*

Proof It is easy to check that the Arf-invariant of cusp point is equal to 1. Curves on figures 1 and 2 have one pair of imaginary complex-conjugated cusp points. It is easy to calculate $\beta(\tilde{q_\Delta})$ with the help of perturbation of curve. After the calculation we have a contradiction with A.2.

We shall say that multiplicitive sequence of singular point of curve (for the definition see [12]) is odd if the sum of numbers in every round in sequence is odd (i.e. the intersection number of the exceptional divisor of every σ-process of resolution of singularity and the proper preimage of the curve under this σ-process is odd). Let s_j denote this sum, where j is the number of σ-process. It is easy to see that singular points with odd multiplicitive sequence are even. Let us calculate Arf-invariants of these points.

A.4 *The Arf-invariant of singular points with odd multiplicitive sequence of curve* $\mathbf{C}A$ *is equal to* $\sum_j \frac{s_j^2-1}{8}$

Proof Let us glue with the evident diffeomorphism of boundaries the pair $(D^4, D^4 \cap \mathbf{C}A)$ and the pair $(D^4, D^4 \cap \mathbf{C}A_\epsilon)$, where D^4 is a small ball with the center in the singular point and A_ϵ is a very small perturbation of A. Let (X, C) denote the result of gluing. C realizes the homology class dual to $w_2(X)$, because $X \approx S^4$. Let us resolve by σ-processes the singular point of (X, C). Let (M, S) denote the result of resolution. It is easy to see that the self-intersection number of surface S in manifold M is equal to $\sum_j(-s_j^2)$ and the signature of M is equal to $\sum_j(-1)$ (i.e. to the number of σ-processes multiplied by (-1)). The fact that S is a characteristic surface in M follows from the fact that the multiplicitive sequence of the singular point is odd. It is easy to see that the Arf-invariant of surface S is equal to the Arf-invariant of the singular point. Now A.4 follows from Rokhlin congruence

$$8Arf(M, S) \equiv \sum_j (1 - s_j^2) \pmod{16}$$

B Further generalizations

In this appendix we announce results generalizing Theorem 1, Rokhlin congruence [4] and Kharlamov[13]-Gudkov-Krakhnov[14] congruence for curves on surfaces. Proofs of these results will be published separately.

First consider the absolute case.

Let $\mathbf{R}B$ be a nonsingular real algebraic surface, $\mathbf{C}B$ be its complexification. Let $D : H^*(\mathbf{C}B; \mathbf{Z}_2) \to H_*(\mathbf{C}B; \mathbf{Z}_2)$ be the operator of Poincaré duality.

B.1 *If $Dw_2(\mathbf{C}B) = [\mathbf{R}B]$ then there exists a natural separation of $\mathbf{R}B$ into two closed surfaces B_1 and B_2. This separation is determined by the condition that $B_j, j = 1, 2$ is a characteristic surface in $\mathbf{C}B/conj$. There is congruence*

$$\chi(B_j) \equiv \frac{\chi(\mathbf{R}B) - \sigma(\mathbf{C}B)}{4} + \beta(q|_{H_1(B_j;\mathbf{Z}_2)}) \pmod 8$$

where q is the Guillou-Marin form of surface $\mathbf{R}B$ in $\mathbf{C}B$

We shall call surfaces B_1 and B_2 the surfaces of complex separation.

B.2 (Corollary) *If $Dw_2(\mathbf{C}B) = [\mathbf{R}B]$ and $\mathbf{R}B$ is connected surface then*

$$\chi(\mathbf{R}B) \equiv \sigma(\mathbf{C}B) \pmod{32}$$

B.3 (Corollary) *If $Dw_2(\mathbf{C}B) = [\mathbf{R}B]$ then*

$$\chi(\mathbf{R}B) \equiv \sigma(\mathbf{C}B) \pmod 8$$

From B.1 one can deduce a new congruence for complex orientations of curves on a hyperboloid. To formulate these congruences we shall use the integral calculates based on Euler characteristic (see O.Ya.Viro[15]). Let H be a hyperboloid, A be a nonsingular real curve of type I and bidegree (d, r) in H. If curve $\mathbf{R}A$ supplied with the complex orientation realizes zero in $H_1(\mathbf{R}H; \mathbf{Z}_4)$ then every fixed component of $\mathbf{R}H \setminus \mathbf{R}A$ determines function $ind : \mathbf{R}H \setminus \mathbf{R}A \to \mathbf{Z}_4$ – the linking number of point and curve $\mathbf{R}A$ supplied with the complex orientation. Let $\mathbf{R}A$ have components nonshrinking in $\mathbf{R}H$ and these components realize the element with coordinates $(s, t), s, s \geq 0$ in standard basis of $H_1(\mathbf{R}H) \approx H_1(\mathbf{R}P^1 \times \mathbf{R}P^1)$. Let $\mathbf{R}A$ supplied with the complex orientation realize class $l'(s, t), l' \in \mathbf{Z}$ in $H_1(\mathbf{R}H)$.

B.4 *If $l' \equiv 0 \pmod 4, ds + rt \equiv 0 \pmod 4$ then*

$$\int_{\mathbf{R}H} ind^2 d\chi \equiv \frac{dr}{2} \pmod 8$$

B.5 *If $l' \equiv 4 \pmod 8, ds + rt \equiv 2 \pmod 4$ then*

$$\int_{\mathbf{R}H} ind^2 d\chi \equiv \frac{dr}{2} + 4 \pmod 8$$

B.6 *If $l' \equiv 0 \pmod 8$ then*

$$\int_{\mathbf{R}H} ind^2 d\chi \equiv \frac{dr}{2} \pmod 8$$

B.7 *If* $l' \equiv 0 \pmod 8, ds + rt \equiv 0 \pmod 4$ *then*

$$\int_{\mathbf{R}H} ind^2 d\chi \equiv \frac{dr}{2} \pmod{16}$$

Let us now consider the relative case. Let $\mathbf{R}A$ be a nonsingular real algebraic curve in $\mathbf{R}B$, $\mathbf{C}A$ be the complexification of $\mathbf{R}A$. Let e_A denote the normal Euler number of $\mathbf{C}A$ in $\mathbf{C}B$.

B.8 *If* $Dw_2(\mathbf{C}B) + [\mathbf{R}B] + [\mathbf{C}A] = 0$ *then there exists a natural separation of* $\mathbf{R}B \setminus \mathbf{R}A$ *into two surfaces* B_1 *and* B_2*,* $\partial B_1 = \partial B_2 = \mathbf{R}A$*. This separation is determined by the condition that* $\mathbf{C}A/conj \cup B_j$ *is a characteristic surface in* $\mathbf{C}B/conj$*. There is Guillou-Marin form* q_j *on* $H_1(\mathbf{C}A/conj \cup B_j; \mathbf{Z}_2)$ *and there is congruence*

$$\chi(B_j) \equiv \frac{e_A}{4} + \frac{\chi(\mathbf{R}B) - \sigma(\mathbf{C}B)}{4} + \beta(q_j) \pmod 8$$

B.9 *If* $Dw_2(\mathbf{C}B) + [\mathbf{R}B] + [\mathbf{C}A] = 0$ *then difference* $q_j(x) - q(x), x \in H_1(B_j; \mathbf{Z}_2)$ *is equal to the linking number of* x *and* $\mathbf{C}A$ *in* $\mathbf{C}B$*, where* q *is the form from B.1 and* q_j *is the form from B.8*

The following result is an application of B.8 and B.9 to curves on a hyperboloid. Let B_+ be one of two parts of hyperboloid $\mathbf{R}H$ bounded by curve $\mathbf{R}A$ in the case when $d \equiv r \equiv 0 \pmod 2$.

B.10 *Let* $d \equiv r \equiv 0 \pmod 2$ *and* $\frac{d}{2}t + \frac{r}{2}s + s + t \equiv 1 \pmod 2$*.*

- **a)** *If* A *is an M-curve then* $\chi(B_+) \equiv \frac{dr}{2} \pmod 8$
- **b)** *If* A *is an (M-1)-curve then* $\chi(B_+) \equiv \frac{dr}{2} \pm 1 \pmod 8$
- **c)** *If* A *is an (M-2)-curve and* $\chi(B_+) \equiv \frac{dr}{2} + 4 \pmod 8$ *then* A *is of type I*
- **d)** *If* A *is of type I then* $\chi(B_+) \equiv 0 \pmod 4$

Points a) and b) of B.10 in the case when $\frac{d}{2}t + \frac{r}{2}s \equiv 0 \pmod 2, s + t \equiv 1 \pmod 2$ were proved by S.Matsuoka [16] in another way (using 2-sheeted branched coverings of hyperboloid). Point d) of B.10 is a corollary of the modification of Rokhlin formula of complex orientations for modulo 4 case.

The following theorem is a generalization of Rokhlin and Kharlamov-Gudkov-Krakhnov congruences for curves on surfaces, because if B is an M-surface then $Dw_2(\mathbf{C}B) = [\mathbf{R}B]$. Let surface $\mathbf{C}B$ be the transversal intersection of hypersurfaces in $\mathbf{C}P^q$ defined by equations $P_j(x_0, \dots, x_q) = 0, j = 1, \dots, s-1$. Let curve $\mathbf{C}A$ be the transversal intersection of $\mathbf{C}B$ and the hypersurface in $\mathbf{C}P^q$ defined by equation $P_s(x_0, \dots, x_q) = 0$, where P_j is a real homogeneous polynomial, $degP_j = m_j$, $q = s + 1$. Let d denote $rk(in_* : H_1(B_+; \mathbf{Z}_2) \to H_1(\mathbf{R}B; \mathbf{Z}_2))$.

B.11 *Let* B *be of type I abs in the case when* $\sum_{j=1}^{s-1} = s \pmod 2$ *or type I rel in the case when* $\sum_{j=1}^{s-1} = s + 1 \pmod 2$ *(for the definitions of type of surfaces see [1]). Let* B_+ *lie completely in one surface of complex separation. Let homomorphism* $inj_* : H_1(\mathbf{R}A; \mathbf{Z}_2) \to H_1(\mathbf{R}B; Z_2)$ *be equal to zero. Let* A *be an (M-k)-curve. Let* m_s *be even and let, in the case when* $m_s \equiv 2 \pmod 4$*, surface* B_- *is contractible in* $\mathbf{R}P^q$*.*

a) *If* $d+k=0$ *then* $\chi(B_+) \equiv \frac{m_1 \ldots m_{s-1} m_s^2}{4} \pmod 8$

b) *If* $d+k=1$ *then* $\chi(B_+) \equiv \frac{m_1 \ldots m_{s-1} m_s^2}{4} \pm 1 \pmod 8$

c) *If* $d+k=2$ *and* $\chi(B_+) \equiv \frac{m_1 \ldots m_{s-1} m_s^2}{4} + 4 \pmod 8$ *then* A *is of type I and* B_+ *is orientable*

d) *If* A *is of type I and* B_+ *is orientable then* $\chi(B_+) \equiv \frac{m_1 \ldots m_{s-1} m_s^2}{4} \pmod 4$

Next theorem allows to calculate the $\mathbf{Z}_2$-Netsvetaev class (see [1]) for regular complete intersections. It is easy to see that $\mathbf{C}B$ is spin manifold iff $\sum_j (m_j - 1) \equiv 1 \pmod 2$. Denote by $\alpha \in H^1(\mathbf{R}B; \mathbf{Z}_2)$ the class induced under the inclusion map : $\mathbf{R}B \to \mathbf{R}P^q$ from the only nontrivial element of $H^1(\mathbf{R}P^q; \mathbf{Z}_2)$.

B.12 *If* $\sum_j (m_j - 1) \equiv 1 \pmod 2$ *then the* $\mathbf{Z}_2$*-Netsvetaev class is well-defined and equal to*

$$\frac{1}{2}(1 + \sum_j (m_j - 1))\alpha \in H^1(\mathbf{R}B; \mathbf{Z}_2)$$

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